\documentclass{article}

\usepackage[utf8]{inputenc}

\usepackage{graphicx}

\usepackage{calligra}

\usepackage{amsmath}

\DeclareMathAlphabet{\mathcalligra}{T1}{calligra}{m}{n}

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\newcommand{\scripty}[1]{\ensuremath{\mathcalligra{#1}}}

\begin{document}

\begin{titlepage}

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\textsc{\huge Examining Radiation of a Four Dipole System}\newline

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{\large \bfseries SUNY Polytechnic Institute}\\

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\textsc{\large Electrical Engineering/Physics}

\newline\large Spring 2018\newline

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\textsc{\LARGE \bf Abstract}\\\\

This paper studies a system of four eclectic dipoles and the radiation that the system produces. The radiation patern for this system will be found by first analyzing the scalar and vector potentials caused by the dipoles and furthermore examine the electric and magnetic fields that the dipoles produce. The Poynting vector and intensity profile for the system will also be examined. The system contains two dipoles of opposite charge on the vertical z-axis and two oppositely charged dipoles on the horizontal y-axis which has a very symmetric layout. In order to build up the techniques needed to examine the four dipole system, this paper will analyze the radiation produced by a single dipole and also the radiation produced by a system of two oppositely charged dipoles on the vertical z-axis. The radiation plots of each system will be compared and analyzed as well.

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\section{Introduction}

An electric dipole is a separation of charges of the opposite sign with the same absolute value separated by a very small distance. When these charges accelerate it creates changing currents which can produce electromagnetic waves. The energy created by these fields that continue to infinity is called radiation. As we know, electromagnetic waves propagate from their origin out to infinity. In this paper, we will be examining a point that will be effected by the radiation of four electric dipoles. Because we are dealing with a point a significant distance away from the electric dipoles we will have to consider retarded times and further more the retarded potentials and fields.

\subsection{Single Electric Dipole Radiation}

Before we can examine a system of four electric dipoles, we must understand how a single electric dipole radiates. Consider the figure below (Figure 1).

\begin{figure}[h]

\includegraphics[width=\textwidth]{Single\_dipole\_diagram.PNG}

\caption{Single Dipole Diagram}

\label{fig:single\_dipole\_diagram}

\end{figure}

The figure shows that there are two charges that are separated by a distance of d and we can imagine that they are connected by a very small and thin wire in which the the charge moves back and forth at and angular frequency \(\omega\). We can say that the charge is

\begin{equation}

q(t) = q\_0\cos(\omega t)

\end{equation}

which when oscillating will yield an electric dipole

\begin{equation}

\vec{p}(t) = p\_0\cos(\omega t)\hat{z}

\end{equation}

where \(p\_0 = q\_0 d\) is the maximum value of the dipole moment. We can now use the generalized equations for retarded potentials which can be seen below

\begin{equation}

V(\vec{r},t) = \frac{1}{4\pi\epsilon\_0}\int\_{a}^{b}\frac{\rho(\vec{r'}, t\_r)}{\scripty{r}}d\tau'

\end{equation}

and

\begin{equation}

\vec{A}(\vec{r},t) = \frac{\mu\_0}{4\pi}\int\_{a}^{b}\frac{\vec{J}(\vec{r'}, t\_r)}{\scripty{r}}d\tau'

\end{equation}

Plugging into the equation for retarded potential we get the following

\begin{equation}

V(\vec{r},t) = \frac{1}{4\pi\epsilon\_0}\left[{\frac{q\_0\cos[\omega(t-\scripty{r}\_+/c)]}{\scripty{r}\_+}}-\frac{q\_0\cos[\omega(t-\scripty{r}\_-/c)]}{\scripty{r}\_-}\right]

\end{equation}

We can now apply the law of cosines to get the equations

\begin{equation}

\scripty{r}\_\pm = \sqrt{r^2\mp rd\cos\theta+(d/2)^2}

\end{equation}

To make this dipole into a perfect dipole, the distance between the two charges must be very small which means we will make the approximation that \(d << r\). If $d$ were to be zero there would be no potential at all so we will carry an expansion to the first order in $d$. Thus we can say

\begin{equation}

\scripty{r}\_\pm \approx r(1\mp \frac{d}{2r}\cos\theta)

\end{equation}

and therefore

\begin{equation}

\frac{1}{\scripty{r}\_\pm} \approx \frac{1}{r}(1\pm \frac{d}{2r}\cos\theta)

\end{equation}

We will now plug in the script r into \(\cos[\omega(t-\frac{\scripty{r}\pm}{c})]\) which will yield the following:

\begin{equation}

\cos[\omega(t-\frac{\scripty{r}\pm}{c})] \approx \cos[\omega(t-\frac{r}{c}\pm \frac{\omega d}{2c}\cos\theta]

\end{equation}

\begin{equation}

= \cos[\omega(t-\frac{r}{c})]\cos(\frac{\omega d}{2c}\cos\theta)\mp \sin[\omega(t-\frac{r}{c})]\sin(\frac{wd}{2c}\cos\theta).

\end{equation}

In the perfect dipole limit we can further make the approximation that \(d << \frac{c}{\omega}\) which allows us to further simplify equation 10 as

\begin{equation}

\cos[\omega(t-\frac{\scripty{r}\_\pm}{c})] \approx \cos[\omega(t-\frac{r}{c})]\mp \frac{\omega d}{2c}\cos\theta\sin[\omega(t-\frac{r}{c})].

\end{equation}

Now we can plug equations (8) and (11) into equation (5) and get the following

\begin{equation}

V(r,\theta,t) = \frac{p\_0 \cos\theta}{4\pi\epsilon\_0r}\left[{-\frac{\omega}{c}\sin[\omega(t-\frac{r}{c}]+\frac{1}{r}\cos[\omega(t-\frac{r}{c})]}\right]

\end{equation}

Since we only are interested in the fields that survive at long distances in the radiation zone we can make the approximation that \(r >> \frac{c}{\omega}\) which means \(r >> \lambda\). In this region the potential in equation 12 will reduce to

\begin{equation}

V(r,\theta,t)= \frac{-p\_0\omega}{4\pi\epsilon\_0 c}(\frac{\cos\theta}{r})\sin[\omega(t-\frac{r}{c})].

\end{equation}

The vector potential for this dipole can be found by determining the current flowing through the wire which would give

\begin{equation}

\vec{I}(t) = \frac{dq}{dt}\hat{z} = -q\_0\omega\sin(\omega t)\hat{z}

\end{equation}

which plugging into equation (4) will give

\begin{equation}

\vec{A}(\vec{r},t) = \frac{\mu\_0}{4\pi}\int\_{-\frac{d}{2}}^{\frac{d}{2}}\frac{-q\_0\omega\sin[\omega(t-\frac{\scripty{r}}{c})])\hat{z}}{\scripty{r}}dz

\end{equation}

Since the integration produces a factor of d, we can replace the integrand by its value at the center to the first order which will give

\begin{equation}

\vec{A}(r,\theta,t) = -\frac{\mu\_0p\_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c}]\hat{z}

\end{equation}

Now that we have acquired the vector and scalar potentials, we can compute the electric and magnetic fields produced by the dipole. We will use the two equations below:

\begin{equation}

\vec{E}=-\vec{\bigtriangledown} V-\frac{\partial{\vec{A}}}{\partial{t}}

\end{equation}

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A}

\end{equation}

Starting with eclectic field we will solve for the gradient of the scalar potential.

\begin{equation}

\vec{\bigtriangledown}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}

\end{equation}

\begin{equation}

\begin{split}

\vec{\bigtriangledown}V = & \frac{-p\_0\omega}{4\pi\epsilon\_0c}\left[\cos\theta(\frac{-1}{r^2}\sin[\omega(t-\frac{r}{c})]\right. \\ &\left.-\frac{\omega}{rc}\cos[\omega(t-\frac{r}{c})])\hat{r}-\frac{\sin\theta}{r^2}\sin[\omega(t-\frac{r}{c})]\hat{\theta}\right]

\end{split}

\end{equation}

\begin{equation}

\vec{\bigtriangledown} V \approx \frac{p\_0\omega^2}{4\pi\epsilon\_0c^2}(\frac{\cos\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{r}.

\end{equation}

Now for the partial derivative for the vector potential with respect to t.

\begin{equation}

\frac{\partial{\vec{A}}}{\partial{t}} = -\frac{\mu\_0p\_0\omega^2}{4\pi r}\cos[\omega(t-\frac{r}{c})](\cos\theta\hat{r}-\sin\theta\hat{\theta})

\end{equation}

Now we will take the curl of the vector potential.

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A} = \frac{1}{r}\left[ \frac{\partial}{\partial r}(rA\_{\theta}) - \frac{\partial}{\partial \theta}(A\_r) \right]\hat{\phi}

\end{equation}

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A} = -\frac{\mu\_0p\_0\omega}{4\pi r}\left[{\frac{\omega}{c}\sin\theta\cos[\omega(t-\frac{r}{c})]+\frac{\sin\theta}{r}\sin[\omega(t-\frac{r}{c})]}\hat{\phi}\right].

\end{equation}

Putting together the equations we just solved for and dropping terms to satisfy our third approximation, we can finally say

\begin{equation}

\vec{E} = -\vec{\bigtriangledown} V - \frac{\partial{\vec{A}}}{\partial{t}} = -\frac{\mu\_0p\_0\omega^2}{4\pi}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\theta}.

\end{equation}

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A} = -\frac{\mu\_0p\_0\omega^2}{4\pi c}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]\hat{\phi}.

\end{equation}

Now that we have the electric and magnetic fields, we can find the energy radiated by an oscillating electric dipole which will be determined by solving for the pointing vector.

\begin{equation}

\vec{S} = \frac{1}{\mu\_0}(\vec{E}\times\vec{B}) = \frac{\mu\_0}{c}\left[{\frac{p\_0\omega^2}{4\pi}(\frac{\sin\theta}{r})\cos[\omega(t-\frac{r}{c})]}^2\hat{r}\right].

\end{equation}

The intensity of the dipole can by obtained by averaging over a complete cycle in time.

\begin{equation}

\langle\vec{S}\rangle = (\frac{\mu\_0p^2\_0\omega^4}{32\pi^2c})\frac{\sin^2\theta}{r^2}\hat{r}.

\end{equation}

And finally the total power radiated is found by integrating \(<\vec{S}>\) over a sphere of radius $r$.

\begin{equation}

\langle P\rangle = \int\langle \vec{S} \rangle \cdot d\vec{a} = \frac{\mu\_0p^2\_0\omega^4}{32\pi^2c}\int\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta d\phi = \frac{\mu\_0p^2\_0\omega^4}{12\pi c}.

\end{equation}

When we plot the intensity profile (Figure 2) we get the following image.

\begin{figure}[h]

\includegraphics[width=\textwidth]{1\_dipole\_rad\_plot.PNG}

\caption{Single Dipole Radiation Plot}

\label{fig:1\_dipole\_rad}

\end{figure}

One can see that the intensity profile looks like the shape of a donut. We will come back to this image for comparison when examining the intensity profile of more complex systems involved multiple electric dipoles.

\subsection{Two Oppositely Charged Electric Dipoles}

Now that we have looked at the radiation pattern for a single electric dipole, we will observe a system of two electric dipoles so that we can understand how symmetry will affect the radiation. When solving this system we will still use the same approximations as before which were:

\textbf{Approximation 1:}

\begin{equation}

d << r

\end{equation}

\textbf{Approximation 2:}

\begin{equation}

d << c/\omega

\end{equation}

\textbf{Approximation 3:}

\begin{equation}

r >> c/\omega

\end{equation}

We will use these approximations and what we have learned from finding the radiation of a single electric dipole to solve for a system of two electric dipoles along the same vertical axis. A figure of the system can be seen below (Figure 3).

\begin{figure}

\includegraphics[width=\textwidth]{single-dipole-system.JPG}

\caption{System of Two Electric Dipoles}

\label{fig:single\_dipole}

\end{figure}

We will use many of the same equations that were used in section 3 with a few small changes. First let us solve for the potentials. In this system the potentials are now affected by two different dipoles so we will find the potentials of each individually and add them together to get our total potentials. The first step in solving for the potentials is establishing our distances and angles. We will start by doing the following.

\begin{equation}

\scripty{r}^2\_\pm = r^2 + (\frac{d}{2})^2 \mp 2r(\frac{d}{2})\cos\theta

\end{equation}

\begin{equation}

\scripty{r}\_\pm = \sqrt{r^2 + (\frac{d}{2})^2 \mp 2r(\frac{d}{2})\cos\theta}

\end{equation}

\begin{equation}

\scripty{r}\_\pm = r\sqrt{1 \mp (\frac{d}{r}\cos\theta)} = r(1 \mp \frac{d}{2r}\cos\theta)

\end{equation}

Next we will solve for \(\cos\theta\_\pm\).

\begin{equation}

\cos\theta\_\pm = \frac{r\cos\theta\mp\frac{d}{2}}{\scripty{r}\_\pm} = r(\cos\theta\mp\frac{d}{2r})\frac{1}{r}(1\pm\frac{d}{2r}\cos\theta)

\end{equation}

Simplifying the last equation we get

\begin{equation}

\cos\theta\_\pm = \cos\theta\pm\frac{d}{2r}\cos^2\theta\mp\frac{d}{2r} = \cos\theta\mp\frac{d}{2r}(1-\cos^2\theta)

\end{equation}

Which allows us to finally say

\begin{equation}

\cos\theta\_\pm = \cos\theta \mp \frac{d}{2r}\sin^2\theta.

\end{equation}

Lastly, before plugging into our potential equation we will solve for \(\sin[\omega(t-\frac{\scripty{r}\_\pm}{c})]\).

\begin{equation}

\sin[\omega(t-\frac{\scripty{r}\_\pm}{c})] = \sin[\omega(t-\frac{r}{c}(1\mp\frac{d}{2r}\cos\theta))] = \sin(\omega t\_0 \pm \frac{\omega d}{2c}\cos\theta)

\end{equation}

where \(t\_0 = t-\frac{r}{c}\). Further simplifying this equation we get

\begin{equation}

\sin(\omega t\_0 \pm \frac{\omega d}{2c}\cos\theta) = \sin(\omega t\_0)\cos(\frac{\omega d}{2c}\cos\theta)\pm\cos(\omega t\_0)\sin(\frac{\omega d}{2c}\cos\theta)

\end{equation}

which can be finally written as

\begin{equation}

\sin[\omega(t-\frac{\scripty{r}\_\pm}{c})] = \sin(\omega t\_0)\pm \frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)

\end{equation}

Now we have everything we need to plug into our potential equations. First we will solve for the scalar potential using the following

\begin{equation}

V\_{\pm} = \frac{-p\_0\omega}{4\pi\epsilon\_0c}(\frac{\cos\theta\_{\pm}}{\scripty{r}\_{\pm}})\sin[\omega(t-\frac{\scripty{r}\_{\pm}}{c})]

\end{equation}

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}\left[(1\pm\frac{d}{2r}\cos\theta)(\cos\theta\mp\frac{d}{2r}\sin^2\theta)\left(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)\right)\right]

\end{equation}

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}\left[\left(\cos\theta\mp\frac{d}{2r}\sin^2\theta\pm\frac{d}{2r}\cos^2\theta\right)\left(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)\right)\right]

\end{equation}

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}\left[\cos\theta\sin(

\omega t\_0) \pm \frac{\omega d}{2c}\cos^2\theta\cos(\omega t\_0) \pm \frac{d}{2r}(\cos^2\theta-\sin^2\theta)\sin(\omega t\_0)\right]

\end{equation}

\newline\newline

We will call the sum of these two expressions V.Adding together the two expressions we get

\begin{equation}

V = \frac{-p\_0\omega}{4\pi\epsilon\_0cr}\left[\frac{\omega d}{c}\cos^2\theta\cos(\omega t\_0)+\frac{d}{r}(\cos^2\theta-\sin^2\theta)\sin\omega t\_0)\right]

\end{equation}

\begin{equation}

V = \frac{-p\_0\omega^2 d}{4\pi\epsilon\_0c^2r}\left[\cos^2\theta\cos(\omega t\_0)+\frac{c}{\omega r}(\cos^2\theta-\sin^2\theta)\sin(\omega t\_0)\right]

\end{equation}

And using our third approximation which said \(r >> \omega/c\) (equation 32), we can write V as

\begin{equation}

V = \frac{-p\_0\omega^2 d}{4\pi\epsilon\_0c^2r}[\cos^2\theta\cos(\omega t\_0)]

\end{equation}

and substituting back in \(t\_0 = t-\frac{r}{c}\) we get

\begin{equation}

V = \frac{-p\_0\omega^2d}{4\pi\epsilon\_0c^2r}\cos^2\theta\cos[\omega(t - \frac{r}{c})]

\end{equation}

Next we will solve for our vector potential by plugging into the following

\begin{equation}

\vec{A} = \frac{-\mu\_0p\_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]]

\end{equation}

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi \scripty{r}\_{\pm}}\sin[\omega(t-\frac{\scripty{r}\_{\pm}}{c})]

\end{equation}

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}\left[(1\pm\frac{d}{2r}\cos\theta)[\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)]\right]\hat{z}

\end{equation}

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}\left[(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)\pm\frac{d}{2r}\cos\theta\sin(\omega t\_0)\right]\hat{z}

\end{equation}

Now we can say \(\vec{A} = \vec{A}\_+ + \vec{A}\_-\).

\begin{equation}

\vec{A} = -\frac{\mu\_0p\_0\omega}{4\pi r}\left[\frac{\omega d}{c}\cos\theta\cos(\omega t\_0)+\frac{d}{r}\cos\theta\sin(\omega t\_0)\right]\hat{z}

\end{equation}

Then factoring out \(\frac{\omega d\cos\theta}{c}\) we are get

\begin{equation}

\vec{A} = - \frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta\left[\cos(\omega t\_0)+\frac{c}{\omega r}\sin(\omega t\_0)\right]\hat{z}.

\end{equation}

And again in our radiation zone we can apply our third assumption (equation 32) which will eliminate the second term giving us

\begin{equation}

\vec{A} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta\cos(\omega t\_0)\hat{z}.

\end{equation}

and substituting back in for \(t\_0\)

\begin{equation}

\vec{A} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta\cos[\omega(t-\frac{r}{c})]\hat{z}.

\end{equation}

As before, we will use these potentials to solve for the electric and magnetic fields produced by the dipoles. We will use the same equations as before which are again shown below

\begin{equation}

\vec{E}=-\vec{\bigtriangledown} V-\frac{\partial{\vec{A}}}{\partial{t}}

\end{equation}

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A}

\end{equation}

We will start by first solving for the gradient of the scalar potential.

\begin{equation}

\begin{split}

\vec{\bigtriangledown} V =& \frac{-p\_0\omega^2d}{4\pi\epsilon\_0c^2}\cos^2\theta(\frac{\omega r \sin[[\omega(t-\frac{r}{c})]-\cos[\omega(t-\frac{r}{c})]}{r^2})\hat{r} \\ + &\frac{p\_0\omega^2d}{2\pi\epsilon\_0c^2r^2}\cos(\theta)\sin(\theta)\cos[\omega(t-\frac{r}{c})]\hat{\theta}

\end{split}

\end{equation}

And using the approximation that \(r >> \frac{c}{\omega}\) we can say

\begin{equation}

\vec{\bigtriangledown} V = \frac{-p\_0\omega^3d}{4\pi\epsilon\_0c^3}\frac{\cos^2(\theta)}{r}\sin[\omega(t-\frac{r}{c})]\hat{r}

\end{equation}

Lastly, we will solve for the partial derivative of the vector potential with respect to time.

\begin{equation}

\frac{\partial{\vec{A}}}{\partial{t}} = \frac{\mu\_0p\_0\omega^3}{4\pi rc}\cos\theta\sin[\omega(t-\frac{r}{c})](\cos\theta\hat{r}-\sin\theta\hat{\theta})

\end{equation}

Finally, we can plug into our field equations giving us:

\begin{equation}

\vec{E} = \frac{\mu\_0p\_0\omega^3d}{4\pi cr}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\theta}

\end{equation}

\begin{equation}

\vec{B} = \frac{\mu\_0p\_0\omega^3d}{4\pi c^2r}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\phi}

\end{equation}

Now that we have our field vectors, we can solve for the Poynting vector

\begin{equation}

\vec{S} = \frac{1}{\mu\_0}(\vec{E}\times\vec{B})

\end{equation}

\begin{equation}

\vec{S} = \frac{1}{\mu\_0}\left[\frac{-\mu\_0p\_0\omega^3d}{4\pi rc}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\right]^2\hat{r}

\end{equation}

Averaging this vector over a complete cycle in time we get the intensity to be

\begin{equation}

\langle \vec{S} \rangle = \frac{1}{2\mu\_0c}\left[\frac{-\mu\_0p\_0\omega^3d}{4\pi rc}\sin\theta\cos\theta\right]^2

\end{equation}

Finally we can get the total radiated power by integrating the intensity over a sphere of radius r

\begin{equation}

P=\int\langle \vec{S} \rangle\cdot d\vec{a}

\end{equation}

\begin{equation}

P=\frac{1}{\mu\_0}\left[\frac{-\mu\_0p\_0\omega^3d}{4\pi c}\right]^2\int\sin^2\theta\cos^2\theta\sin\theta d\theta d\phi

\end{equation}

Which will finally give us

\begin{equation}

P=\frac{\mu\_0p\_0^2d^2\omega^6}{60\pi c^3}

\end{equation}

The radiation plot for this two dipole system can be seen in Figure 4:

\begin{figure}

\includegraphics[width=\textwidth]{2\_dipole\_rad\_plot.PNG}

\caption{Two Dipole System Radiation Plot}

\label{fig:2\_dipole\_rad}

\end{figure}

One can see that this radiation plot has qualities similar to the plot of just one electric dipole. The single dipole had a circular donut shape to it and this system of two produces two donut shapes that stretch into the space in between the two dipoles making an hourglass shape.

\section{Electric Dipole System: Part I (yz-Plane)}

Now we will solve for the distances to point P from the electric dipoles an the y-axis as shown in the horizontal system in Figure 6.

\begin{figure}

\includegraphics[width=\textwidth]{horizontal-dipole.JPG}

\caption{System of two horizontal electric dipoles}

\label{fig:horizontal-dipoles}

\end{figure}

The dipoles in this system are separated by a distance l. First we will solve for the one labeled \(r\_R\). We will start off again by applying the law of cosines giving us

\begin{equation}

r\_R^2 = r^2+\frac{l}{2}^2-2r\frac{l}{2}\theta'

\end{equation}

where \(\theta'\) is equal to \(\frac{\pi}{2}-\theta\). By taking the square root of both sides we get

\begin{equation}

r\_R = \sqrt{r^2+\frac{l}{2}^2-2r\frac{l}{2}\cos\theta'}

\end{equation}

Similar to \(r\_+\) we know our result will be

\begin{equation}

r\_R = r(1-\frac{l}{2r}\cos\theta')

\end{equation}

but plugging in for \(\theta'\) we will get \(\cos(\frac{\pi}{2}-\theta)\) which we know equals \(sin(\theta)\). Therefor, \(r\_R\) will read

\begin{equation}

r\_R = r(1-\frac{1}{2r}\sin\theta)

\end{equation}

which means that

\begin{equation}

\frac{1}{r\_R} = \frac{1}{r}(1+\frac{l}{2r}\sin\theta)

\end{equation}

Now to look at \(r\_L\). Applying the law of cosines again we get

\begin{equation}

r\_L^2 = r^2+\frac{l}{2}^2-2r\frac{l}{2}\cos\theta''

\end{equation}

where \(\theta''\) is equal to \(\theta+\frac{\pi}{2}\). By taking the square root of both sides we get

\begin{equation}

r\_L = \sqrt{r^2+\frac{l}{2}^2-2r\frac{l}{2}\cos\theta}

\end{equation}

Similar to \(r\_R\) we know our result will be

\begin{equation}

r\_L = r(1-\frac{l}{2r}\cos\theta'')

\end{equation}

but plugging in for \(\theta''\) we will get \(\cos(\theta+\frac{\pi}{2})\) which we know equals \(-\sin\theta)\). Therefor, \(r\_L\) will read

\begin{equation}

r\_L = r(1+\frac{l}{2r}\sin\theta)

\end{equation}

which means that

\begin{equation}

\frac{1}{r\_L} = \frac{1}{r}(1-\frac{1}{2r}\sin\theta).

\end{equation}

In summary, we have solved for all of the distances from the electric dipoles to the point P. The four equations are restated below:

\begin{equation}

\frac{1}{r\_+} = \frac{1}{r}(1+\frac{d}{2r}\cos\theta)

\end{equation}

\begin{equation}

\frac{1}{r\_-} = \frac{1}{r}(1-\frac{d}{2r}\cos\theta)

\end{equation}

\begin{equation}

\frac{1}{r\_R} = \frac{1}{r}(1+\frac{l}{2r}\sin\theta)

\end{equation}

\begin{equation}

\frac{1}{r\_L} = \frac{1}{r}(1-\frac{l}{2r}\sin\theta)

\end{equation}

By solving for the distances in the form of \(\frac{1}{r}\) will allow us to plug them directly into the scalar and vector potential equations.Now that we have solved for each one of these distance we want to plug into the general equation for scalar potential which is

\begin{equation}

V(r,\theta,t)=\frac{-p\_0\omega}{4\pi\epsilon\_0c}\frac{\cos\theta}{r}\sin[\omega(t-\frac{r}{c})]

\end{equation}

where r is each r solved above and each \(\cos\theta\) is from the corresponding \(\theta\) values shown in Figure 1. So now let us solve for each \(\cos\theta\). First we will solve for \(\cos\theta\_+\). Lets say

\begin{equation}

\cos\theta\_+ = \frac{h\_+}{r\_+} = (r\cos\theta-\frac{d}{2})(\frac{1}{r})(1+\frac{d}{2r}\cos\theta)

\end{equation}

\begin{equation}

\cos\theta\_+ = \cos\theta+\frac{d}{2r}\cos^2\theta-\frac{d}{2r}-\frac{d^2}{4r^2}\cos\theta

\end{equation}

Remembering our first assumption where \(d<<r\) we can say

\begin{equation}

\cos\theta\_+ = \cos\theta+\frac{d}{2r}\cos^2\theta-\frac{d}{2r}

\end{equation}

\begin{equation}

\cos\theta\_+ = \cos\theta-\frac{d}{2r}(1-\cos^2\theta)

\end{equation}

which will give us

\begin{equation}

\cos\theta\_+ = \cos\theta-\frac{d}{2r}\sin^2\theta.

\end{equation}

Similarly, for \(\cos\theta\_-\) we will get

\begin{equation}

\cos\theta\_- = \cos\theta+\frac{d}{2r}\sin^2\theta.

\end{equation}

We will do a similar process for the dipoles on the y-axis. Lets look at \(\cos\theta\_R\) first.

\begin{equation}

\cos\theta\_R = \frac{h\_R}{r\_R} = (r\sin\theta-\frac{l}{2})(\frac{1}{r})(1+\frac{l}{2r}\sin\theta)

\end{equation}

\begin{equation}

\cos\theta\_R = sin\theta+\frac{l}{2r}\sin^2\theta-\frac{l}{2r}-\frac{l^2}{4r^2}\sin\theta

\end{equation}

And like before the \(\frac{l^2}{4r^2}\sin\theta\) will drop due to our assumptions leaving us with

\begin{equation}

\cos\theta\_R = \sin\theta+\frac{l}{2r}\sin^2\theta-\frac{l}{2r}

\end{equation}

\begin{equation}

\cos\theta\_R = \sin\theta-\frac{l}{2r}(1-\sin^2\theta)

\end{equation}

Finally giving us

\begin{equation}

\cos\theta\_R = \sin\theta-\frac{l}{2r}\cos^2\theta.

\end{equation}

Looking at \(\cos\theta\_L\) we have an equation similar to \(cos\theta\_R\) with the opposite sign giving us

\begin{equation}

\cos\theta\_L = \sin\theta+\frac{l}{2r}\cos^2\theta.

\end{equation}

Now we have all four equations for \(\cos\theta\) (equations 35, 36, 41, and 42). They are shown again below

\[\cos\theta\_+ = \cos\theta-\frac{d}{2r}\sin^2\theta.\]

\[\cos\theta\_- = \cos\theta+\frac{d}{2r}\sin^2\theta.\]

\[\cos\theta\_R = \sin\theta-\frac{l}{2r}\cos^2\theta.\]

\[\cos\theta\_L = -\sin\theta-\frac{l}{2r}\cos^2\theta.\]

Now we are ready to plug into the general equation to get the scalar potential contribution for each electric dipole. Plugging the parts in for each dipole we are given these four equations

\begin{equation}

V\_+=\frac{-p\_0\omega}{4\pi\epsilon\_0c}\left(\frac{\cos\theta-\frac{d}{2r}\sin^2\theta}{r(1-\frac{d}{2r}\cos\theta)}\right)\sin\left[\omega\left(t-\frac{r(1-\frac{d}{2r}\cos\theta)}{c}\right)\right]

\end{equation}

\begin{equation}

V\_-=\frac{p\_0\omega}{4\pi\epsilon\_0c}\left(\frac{\cos\theta+\frac{d}{2r}\sin^2\theta}{r(1+\frac{d}{2r}\cos\theta)}\right)\sin\left[\omega\left(t-\frac{r(1+\frac{d}{2r}\cos\theta)}{c}\right)\right]

\end{equation}

\begin{equation}

V\_R=\frac{-p\_0\omega}{4\pi\epsilon\_0c}\left(\frac{\sin\theta-\frac{l}{2r}\cos^2\theta}{r(1-\frac{1}{2r}\sin\theta)}\right)\sin\left[\omega\left(t-\frac{r(1-\frac{1}{2r}\sin\theta)}{c}\right)\right]

\end{equation}

\begin{equation}

V\_L=\frac{p\_0\omega}{4\pi\epsilon\_0c}\left(\frac{-\sin\theta-\frac{l}{2r}\cos^2\theta}{r(1+\frac{1}{2r}\sin\theta)}\right)\sin\left[\omega\left(t-\frac{r(1+\frac{1}{2r}\sin\theta)}{c}\right)\right]

\end{equation}

In order to get the total potential at point P, we need to add together the potentials of each dipole. However, in order to add of of these equations, it will be easier to go back to the original form seen again below (Eq. 42) and add them together two at a time.

\[V = \frac{-p\_0\omega}{4\pi\epsilon\_0c}(\frac{\cos\theta}{r})\sin[\omega(t-\frac{r}{c})]\]

The easiest way to solve for the potentials in this format is to solve for \(\sin[\omega(t-\frac{r\_{\pm}}{c})]\) and \(\cos\theta\_{\pm}\) first. Once we have solved for these pieces, we can plug into the general equation that was shown above in equation blank. This will allow us to solve for the contribution of potential for dipoles on the x axis. We will have to repeat these calculations in order to get \(\sin[\omega(t-\frac{r\_{RL}}{c})]\) and \(\cos\theta\_{RL}\) which will allow us to solve for the contribution of potential from the dipoles on the y axis.

\begin{equation}

\sin[\omega(t-\frac{r\_{\pm}}{c})] = \sin\left[\omega\left(t-\frac{r}{c}(1\pm\frac{d}{2r}\cos\theta)\right)\right]

\end{equation}

where

\begin{equation}

\sin\left[\omega\left(t-\frac{r}{c}\mp\frac{d}{2c}\cos\theta\right)\right] = \sin\left(\omega t\_0\mp\frac{d\omega}{2c}\cos\theta\right)

\end{equation}

Then we could say

\begin{equation}

\sin[\omega(t-\frac{r\_{\pm}}{c})]=\sin\left[\omega t\_0 \pm \frac{\omega d}{2c}\cos\theta\right]

\end{equation}

where \(t\_0 = t-\frac{r}{c}\). Now using the identity:

\begin{equation}

\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)\pm\cos(\alpha)\sin(\beta)

\end{equation}

we can write the equation below as

\begin{equation}

\sin\left[\omega t\_0 \pm \frac{\omega d}{2c}\cos\theta\right] = \sin(\omega t\_0)\cos\left(\frac{\omega d}{2c}\cos\theta\right)\pm\cos(\omega t\_0)\sin\left[\frac{\omega d}{2c}\cos\theta\right]

\end{equation}

We can simplify this further by saying

\begin{equation}

\cos\left(\frac{\omega d}{2c}\cos\theta\right) = 1

\end{equation}

and

\begin{equation}

\sin[\frac{\omega d}{2c}\cos\theta] = \frac{\omega d}{2c}\cos\theta

\end{equation}

Now we can finally write

\begin{equation}

\sin\left[\omega t\_0 \pm \frac{\omega d}{2c}\cos\theta\right] = \sin(\omega t\_0) \pm \frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)

\end{equation}

Not lets simplify \(\cos\theta\_{\pm}\).

\begin{equation}

\cos\theta\_{\pm} = \frac{r\cos\theta \mp \frac{d}{2}}{r\_{\pm}} = r(\cos\theta \mp \frac{d}{2r})(\frac{1}{r})(1 \pm \frac{d}{2r}\cos\theta)

\end{equation}

when multiplied together will give us

\begin{equation}

\cos\theta \pm \frac{d}{2r}\cos^2\theta \mp \frac{d}{2r} = \cos\theta \mp \frac{d}{2r}(1-\cos^2\theta) = \cos\theta \mp \frac{d}{2r}\sin^2\theta

\end{equation}

So we can finally say

\begin{equation}

\cos\theta\_{\pm} = \cos\theta \mp \frac{d}{2r}\sin^2\theta

\end{equation}

Using these identities, we can write out potentials in a form that is easier to combine. Lets start by looking at \(V\_+\) and \(V\_-\) (the potentials on the y axis). We can write them as follows:

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[(1\pm\frac{d}{2r}\cos\theta)(\cos\theta\mp\frac{d}{2r}\sin^2\theta)(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0))]

\end{equation}

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[(\cos\theta\mp\frac{d}{2r}\sin^2\theta\pm\frac{d}{2r}\cos^2\theta)(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0))]

\end{equation}

\begin{equation}

V\_{\pm} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[\cos\theta\sin(

\omega t\_0) \pm \frac{\omega d}{2c}\cos^2\theta\cos(\omega t\_0) \pm \frac{d}{2r}(\cos^2\theta-\sin^2\theta)\sin(\omega t\_0)]

\end{equation}

\newline\newline

We will call the sum of these two expressions \(V\_{T1}\).Adding together the two expressions we get

\begin{equation}

V\_{T1} = \frac{-p\_0\omega}{4\pi\epsilon\_0cr}[\frac{\omega d}{c}\cos^2\theta\cos(\omega t\_0)+\frac{d}{r}(\cos^2\theta-\sin^2\theta)\sin\omega t\_0)]

\end{equation}

\begin{equation}

V\_{T1} = \frac{-p\_0\omega^2 d}{4\pi\epsilon\_0c^2r}[\cos^2\theta\cos(\omega t\_0)+\frac{c}{\omega r}(\cos^2\theta-\sin^2\theta)\sin(\omega t\_0)]

\end{equation}

And using our third approximation which said \(r >> \omega/c\) (equation 3), we can write \(V\_{T1}\) as

\begin{equation}

V\_{T1} = \frac{-p\_0\omega^2 d}{4\pi\epsilon\_0c^2r}[\cos^2\theta\cos(\omega t\_0)]

\end{equation}

and subbing back in \(t\_0 = t-\frac{r}{c}\) we get

\begin{equation}

V\_{T1} = \frac{-p\_0\omega^2d}{4\pi\epsilon\_0c^2r}\cos^2\theta\cos[\omega(t - \frac{r}{c})]

\end{equation}

Now we must look at the sum for \(V\_R\) and \(V\_L\). Similar to before, we will start by solving for the \(\sin[\omega(t-\frac{r}{c})]\) piece of our general equation (equation 30). Using our distances \(r\_r\) and \(r\_L\) we can write the following

\begin{equation}

\sin[\omega(t-\frac{r\_{RL}}{c})] = \sin[\omega(t-\frac{r}{c}(1\pm\frac{l}{2r}\sin\theta)]

\end{equation}

where

\begin{equation}

\sin[\omega(t-\frac{r}{c}(1\pm\frac{l}{2r}\cos\theta)] = \sin(\omega t\_0\mp\frac{\omega l}{2c}\sin\theta)

\end{equation}

which allows us to write

\begin{equation}

\sin[\omega(t-\frac{r\_{RL}}{c})] = \sin[\omega t\_0\pm\frac{\omega l}{2c}\sin\theta]

\end{equation}

where \(t\_0 = t -\frac{r}{c}\). Using the same trig identity as before (equation 50) we can write

\begin{equation}

\sin[\omega t\_0 \pm\frac{\omega l}{2c}\sin\theta]=\sin(\omega t\_0)\cos(\frac{\omega l}{2c}\sin\theta)\pm\cos(\omega t\_0)\sin(\frac{\omega l}{2c}\sin\theta)

\end{equation}

where

\begin{equation}

\cos(\frac{\omega l}{2c}\sin\theta) = 1

\end{equation}

and

\begin{equation}

\sin[\frac{\omega l}{2c}\sin\theta] = \frac{\omega l}{2c}\sin\theta

\end{equation}

Now we can finally write

\begin{equation}

\sin[\omega t\_0\pm\frac{\omega l}{2c}\sin\theta]=\sin(\omega t\_0)\pm\frac{\omega l}{2c}\sin\theta\cos(\omega t\_0)

\end{equation}

From earlier, we solved for \(\cos\theta\_{RL}\). We can then plug in for the scalar potential which will yield the equation

\begin{equation}

V\_{RL} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[(1\pm\frac{l}{2r}\sin\theta)(\pm\sin\theta-\frac{l}{2r}\cos^2\theta)(\sin(\omega t\_0)\pm\frac{\omega l}{2c}\sin\theta\cos(\omega t\_0))

\end{equation}

\begin{equation}

V\_{RL} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[(\frac{-l}{2r}\cos^2\theta\pm\sin\theta+\frac{l}{2r}\sin^2\theta)(\sin(\omega t\_0)\pm\frac{\omega l}{2c}\sin\theta\cos(\omega t\_0))

\end{equation}

Which after some multiplication and reduction gives us

\begin{equation}

V\_{RL} = \frac{\mp p\_0\omega}{4\pi\epsilon\_0cr}[(\sin\theta

\sin(\omega t\_0)\pm\frac{wl}{2r}\sin^2\theta\cos(\omega t\_0)]

\end{equation}

We will call the sum of this expression \(V\_{T2}\). After adding the pieces up and subbing back in for \(t\_0\) we get

\begin{equation}

V\_{T2} = \frac{-p\_0\omega^2 l}{4\pi\epsilon\_0 c^2r}\sin^2\theta\cos[\omega(t-\frac{r}{c})]

\end{equation}

Now that we have the contribution of each dipole, we need to add them together to get the total scalar potential at point P.

\begin{equation}

V\_T = V\_++V\_-+V\_R+V\_L

\end{equation}

or

\begin{equation}

V\_T = V\_{T1} + V\_{T2}

\end{equation}

Subbing in for \(V\_{T1}\) and \(V\_{T2}\) we will get

\begin{equation}

V\_T = \frac{-p\_0\omega^2 d}{4\pi\epsilon\_0 c^2r}\cos^2\theta\cos[\omega(t-\frac{r}{c})] + \frac{-p\_0\omega^2 l}{4\pi\epsilon\_0 c^2r}\sin^2\theta\cos[\omega(t-\frac{r}{c})]

\end{equation}

To simplify our math, lets say that \(l = d\) so we will sub all of the l's in our equation which will allow us to combine the terms of the equation into the following

\begin{equation}

V\_T\frac{-p\_0\omega^2d}{4\pi\epsilon\_0c^2r}\cos[\omega(t-\frac{r}{c})]

\end{equation}

Now that we have finished solving for the scalar potential given by the dipoles, we must examine the vector potential given by each. The general form for the vector potential produced by an electric dipole can be seen below.

\begin{equation}

\vec{A} = \frac{-\mu\_0p\_0\omega}{4\pi r}\sin[\omega(t-\frac{r}{c})]

\end{equation}

First, lets examine \(\vec{A}\_{+}\) and \(\vec{A}\_{-}\) for our dipoles on the y axis. We can write them as follows:

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r\_{\pm}}\sin[\omega(t-\frac{r\_{\pm}}{c})]

\end{equation}

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}[(1\pm\frac{d}{2r}\cos\theta)[\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)]]\hat{z}

\end{equation}

\begin{equation}

\vec{A\_{\pm}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}[(\sin(\omega t\_0)\pm\frac{\omega d}{2c}\cos\theta\cos(\omega t\_0)\pm\frac{d}{2r}\cos\theta\sin(\omega t\_0)]\hat{z}

\end{equation}

Now we can say \(\vec{A}\_{T1} = \vec{A}\_+ + \vec{A}\_-\).

\begin{equation}

\vec{A}\_{T1} = -\frac{\mu\_0p\_0\omega}{4\pi r}[\frac{\omega d}{c}\cos\theta\cos(\omega t\_0)+\frac{d}{r}\cos\theta\sin(\omega t\_0)]\hat{z}

\end{equation}

Then factoring out \(\frac{\omega d\cos\theta}{c}\) we are get

\begin{equation}

\vec{A}\_{T1} = - \frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta[\cos(\omega t\_0)+\frac{c}{\omega r}\sin(\omega t\_0)]\hat{z}.

\end{equation}

And again in our radiation zone we can apply our third assumption (equation 3) which will eliminate the second term giving us

\begin{equation}

\vec{A}\_{T1} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta\cos(\omega t\_0)\hat{z}.

\end{equation}

and subbing back in for \(t\_0\)

\begin{equation}

\vec{A}\_{T1} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos\theta\cos[\omega(t-\frac{r}{c})]\hat{z}.

\end{equation}

Next we will look at \(\vec{A}\_{R}\) and \(\vec{A}\_{L}\) for the dipoles on the x axis.

\begin{equation}

\vec{A\_{RL}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r\_{RL}}\sin[\omega(t-\frac{r\_{RL}}{c})]

\end{equation}

\begin{equation}

\vec{A\_{RL}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}[(1\pm\frac{l}{2r}\sin\theta)(\sin(\omega t\_0)\pm\frac{\omega l}{2c}\sin\theta\cos(\omega t\_0))]\hat{z}

\end{equation}

\begin{equation}

\vec{A\_{RL}} = \mp \frac{\mu\_0p\_0\omega}{4\pi r}[(\sin(\omega t\_0)\pm\frac{\omega l}{2c}\sin\theta\cos(\omega t\_0)\pm\frac{l}{2r}\sin\theta\cos(\omega t\_0)]\hat{z}

\end{equation}

Similarly, we can now say

\begin{equation}

\vec{A}\_{T2} = \vec{A}\_R + \vec{A}\_L.

\end{equation}

\begin{equation}

\vec{A}\_{T2} = -\frac{\mu\_0p\_0\omega}{4\pi r}[\frac{\omega l}{c}\sin\theta\cos(\omega t\_0)+\frac{l}{r}\sin\theta\cos(\omega t\_0)]\hat{y}

\end{equation}

Where \(\vec{A\_{T2}}\) is in the \(\hat{y}\) direction (and \(\phi = \pi/2\)). Then factoring out \(\frac{\omega l\sin\theta}{c}\) we are get

\begin{equation}

\vec{A}\_{T2} = -\frac{\mu\_0p\_0\omega^2l}{4\pi rc}\sin\theta[\cos(\omega t\_0)+\frac{c}{\omega r}\cos(\omega t\_0)]\hat{y}.

\end{equation}

And applying our third assumption (equation 3) which will eliminate the second term giving us

\begin{equation}

\vec{A}\_{T2} = -\frac{\mu\_0p\_0\omega^2l}{4\pi rc}\sin\theta[\cos(\omega t\_0)]\hat{y}.

\end{equation}

and subbing back in for \(t\_0\)

\begin{equation}

\vec{A}\_{T2} = -\frac{\mu\_0p\_0\omega^2l}{4\pi rc}\sin\theta\cos[\omega(t-\frac{r}{c})]\hat{y}.

\end{equation}

And finally we can write

\begin{equation}

\vec{A}\_{Total} = \vec{A}\_{T1} + \vec{A}\_{T2}.

\end{equation}

Again, we will say that \(l = d\) allowing us to add the terms of the two vector potentials resulting in the following

\begin{equation}

\vec{A}\_{Total} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\cos[\omega(t-\frac{r}{c})](\sin\theta\hat{y}+\cos\theta\hat{z})

\end{equation}

which is the final result for the vector potential given by the four dipoles. Now that we have solved for both the scalar and vector potentials we can plug them into our equations to find the electric and magnetic fields caused by this system of dipoles.

\subsection{Electric and Magnetic Fields}

Now that we have solved for the vector and scalar potentials for each system at point P in the section above, we can use them to calculate the electric and magnetic fields produced by each dipole at point P. Then we can combine the field calculation results together to find the total electric and magnetic fields for the four dipole system.The general equation for electric and magnetic fields can be seen below.

\begin{equation}

\vec{E}=-\vec{\bigtriangledown}V-\frac{\partial{\vec{A}}}{\partial{t}}

\end{equation}

\begin{equation}

\vec{B}=\vec{\bigtriangledown}\times\vec{A}

\end{equation}

We know from Section 1.2 that the electric and magnetic fields from the vertical dipoles are

\begin{equation}

\vec{E\_1} = \frac{\mu\_0p\_0\omega^3d}{4\pi cr}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\theta}

\end{equation}

\begin{equation}

\vec{B\_1} = \frac{\mu\_0p\_0\omega^3d}{4\pi c^2r}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\phi}

\end{equation}

Now we just have to find the electric and magnetic fields from the horizontal system of dipoles. First let us start off by solving for the gradient of the scalar potential

\begin{equation}

\vec{\bigtriangledown}V = \frac{\partial{V}}{\partial{r}}\hat{r}+\frac{1}{r}\frac{\partial{V}}{\partial{\theta}}\hat{\theta}

\end{equation}

Subbing in \(V\_{T2}\) for V and plugging into Maple we get the equation below

\begin{equation}

\vec{\bigtriangledown}V = -\frac{p\_0\omega^3d}{4\pi\epsilon\_0c^3r}\sin^2\theta\sin[\omega(t-\frac{r}{c})](\cos\theta\hat{r}-\sin\theta\hat{\theta})

\end{equation}

Again, using Maple, we sub in \(\hat{A}\_{T2}\) for \(\hat{A}\) we will solve for the partial derivative of the vector potential

\begin{equation}

\frac{\partial{\vec{A}}}{\partial{t}} = -\frac{\mu\_0p\_0\omega^2d}{4\pi rc}\sin\theta\sin[\omega(t-\frac{r}{c})](\sin\theta\hat{r}+\cos\theta\hat{\theta})

\end{equation}

Putting these pieces together in Maple we can solve for the electric field

\begin{equation}

\vec{E\_2} = -\frac{\mu\_0p\_0\omega^3d}{4\pi c^2r}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\theta}

\end{equation}

Now to find the magnetic field

\begin{equation}

\vec{B} = -\vec{\bigtriangledown}\times\vec{A} = \frac{1}{r}[\frac{\partial{\vec{A\_{\theta}}}}{\partial{t}}-\frac{\partial{\vec{A\_{r}}}}{\partial{\theta}}]\hat{\phi}

\end{equation}

Using Maple we get

\begin{equation}

\vec{B\_2} = \frac{\mu\_0p\_0\omega^3d}{4\pi c^2r}\sin\theta\cos\theta\sin[\omega(t-\frac{r}{c})]\hat{\phi}

\end{equation}

Combining together our fields from the vertical and horizontal dipoles, we can get our total electric and magnetic fields produced by the entire system of the four dipoles.

\begin{equation}

\vec{E\_{Total}} = 0

\end{equation}

\begin{equation}

\vec{B\_{Total}} = 0

\end{equation}

As shown above, the dipoles go to zero when only examined on the yz-axis. This can also be seen when examining the polar plots of the two subsystems. The plots of both of the plots can be seen below in Figure 9. These plots look identical which makes sense because the system has symmetry along the lines \(y = z\) and \(y = -z\).

\begin{figure}[h]

\includegraphics[width=\textwidth]{Polar\_Comparison.PNG}

\caption{Polar Plot Comparison}

\label{fig:polar\_comp}

\end{figure}

One can also see how why the fields cancel each other out when examining the direction vectors for each angle describing point P (see Figure 10 below). This diagram shows two angles: \(\theta\) and \(\theta'\). \(\theta\) is the original angle that is used to describe the position of point P with respect to the vertical dipole system. The result for just the vertical dipoles using this angle was found in Section 1.2. If we take this system of dipoles and rotate it 90 degrees so that they fall on the horizontal axis, we should expect the same solution except with respect to the angle \(\theta'\) (shown in Figure 10). However, when examining the direction vectors for the angles, one can see that \(\hat{\theta}\) and \(\hat{\theta}'\) are pointing in exactly opposite direction 180 degrees from each other. Since the direction vectors are perfectly opposite from each other and the magnitudes of the field calculations will be identical, we know that each electric field and magnetic field from the rotated system (the horizontal system that uses \(\theta'\)) will pick up a negative sign which will cause the fields to cancel out when added together.

\begin{figure}[h]

\includegraphics[width=\textwidth]{Theta\_hat\_comp.PNG}

\caption{Theta Hat Comparison}

\label{fig:theta\_hat\_comp}

\end{figure}

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\subsection{Radiated Power}

When examining on just the yz-plane, the Poynting Vector and the radiated power are both zero due to the electric and magnetic fields both having no magnitude. Although each system has its own fields and radiation profiles, when the systems are combined, the fields cancel out since the fields are in opposite directions and they carry the same magnitude when isolated to the yz-plane (with \(\phi = \pi/2\)) due to the symmetry along the lines \(y = z\) and \(y = -z\). A continuation of this project will be made that will explore the radiation pattern when the dipoles are observed everywhere and not just on the yz-plane.

\section{Conclusion}

The calculations done for the four dipole system were completed by using the same methods to compute the radiation of a single dipole. These techniques worked in concert with symmetry to allow us to solve for the four dipole system. These radiation plots were created with respect to only the y-z axis which is why the radiation plots didn't seem to the same plot rotated by 90 degrees when examining the vertical and horizontal dipoles. An extension of this paper will be created to examine the radiation plots with respect to different planes and to go deeper into the symmetrical properties of dipole radiation and expand these solutions into all three dimension, not just the yz-plane.

\section{References}

Griffiths, D. J. (1999) Introduction to Electrodynamics Third Edition. Upper Saddle River, New Jersey: Prentice-Hall.

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section{Appendix}

subsection{Maple Code}

(put maple code here used for symbolic computation)

subsection{Poynting Vector}

Next lets look at the energy radiated by the electric dipoles at point P. To do this, we will need to find the Poynting vector which represents the directional energy flux. The general formula for the Poynting Vector can be seen below.

\begin{equation}

\vec{S}=\frac{1}{\mu\_0}(\vec{E}\times\vec{B})

\end{equation}

Plugging in the total electric and magnetic fields calculated for the system into Maple we get the Poynting vector to be

\begin{equation}

\vec{S} = \frac{\mu\_0p^2\_0\omega^6d^2}{8\pi^2 c^3r^2}\sin^2\theta\cos\theta(\cos\theta+\sin\theta)\sin[\omega(t-\frac{r}{c})]\hat{r}

\end{equation}

subsection{Radiated Power}

Now lets examine the total radiated power by the system at point P. We will do this by integrating the average intensity using the Poynting vector which was found in Section 3.3.

\begin{equation}

P=\int\_{a}^{b}<\vec{S}>\cdot d\vec{a}

\end{equation}

In order to find \(<\vec{S}>\) we will need to average the Poynting vector over one cycle. Using Maple to average one cycle in time we get the following

\begin{equation}

<\vec{S}> = \frac{\mu\_0p^2\_0\omega^6d^2}{16\pi^2 c^3r^2}\sin^2\theta\cos\theta(\cos\theta+\sin\theta)\hat{r}

\end{equation}

Now we can plug this average into our integral over a sphere of radius r with the following set up

\begin{equation}

P=\frac{\mu\_0p^2\_0\omega^6d^2}{16\pi^2 c^3}\int\sin^2\theta\cos\theta(\cos\theta+\sin\theta)\sin\theta d\theta d\phi

\end{equation}

Which yields the result

\begin{equation}

P=\frac{\mu\_0p^2\_0\omega^6d^2}{30\pi c^3}

\end{equation}

The plot of the 4 dipole system radiation profile can be seen below (Figure 10)

\begin{figure}[h]

\includegraphics[width=\textwidth]{4\_dipole\_rad.PNG}

\caption{Four Dipole System Radiation Plot}

\label{fig:4\_dipole\_rad}

\end{figure}

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subsection{Comparing Intensity Profiles}

The figure below (Figure 10) shows all four of the radiation profiles side by side.

\begin{figure}[h]

\includegraphics[width=\textwidth]{dipole\_comparison.PNG}

\caption{Radiation Plot Comparison}

\label{fig:2\_dipole\_rad}

\end{figure}

Figure 10a. shows the radiation of a single dipole from section 1.1. Figure 10b. shows the radiation of a two dipole system of two oppositely positioned dipoles on the vertical z-axis. Figure 10c. shows the radiation of a two dipole system of two oppositely positioned dipoles on the horizontal y-axis. Although this diagram should look similar to 9b. just rotated 90 degrees, it is not the case. This is because when generating the radiation plots, they are only for the y-z plane. To get the radiation plot that looks similar to 10b. the radiation plot would have to be created with respect to the x-y plane. Figure 10d. shows the final radiation plot of the four dipole system.

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